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A quantum mechanical model of spherical supermembranes

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Abstract: We present a quantum mechanical model of spherical supermembranes. Using superfields to represent the cartesian coordinates of the membrane, we are able to exactly determine its supersymmetric vacua. We find there are two classical vacua, one corresponding to an extended membrane and one corresponding to a point-like membrane. For the $\mathcal{N} = 2$ case, instanton effects then lift these vacua to massive states. For the $\mathcal{N} = 4$ case, there is no instanton tunneling, and the vacua remain massless. Similarities to spherical supermembranes as giant gravitons and in Matrix theory on pp-waves is discussed.

Keywords: M(atrix) Theories, Penrose limit and pp-wave background, Supersymmetry Breaking.

Contents

1. Introduction

Spherical supermembranes are a topic which have received much attention lately. First quantized in light-front gauge by de Wit, Hoppe and Nicolai [1], they have become relevant in the study of holography on $AdS_n \times S^m$ spaces [2, 3, 4]. Recent studies of Matrix theory on pp-wave backgrounds has also resulted in studying supermembranes [5, 6, 7, 8], although the results are somewhat obfuscated by the large-component spinors arising from dimensional reduction. Thus a simple model, capturing the essentials of the supermembrane dynamics, would be useful.

In this paper we present a simplified model of spherical supermembranes, based on the quantum mechanics of 3 superfields and 2 or 4 supercharges. In the classical approximation, where quantum-mechanical instanton tunneling effects are negligible, there are 2 supersymmetric vacua present. In the $\mathcal{N} = 2$ case, the vacua have opposite statistics, allowing instanton tunneling. We then identify the instanton responsible for lifting these vacua to massive states. Thus we are left with no supersymmetric vacua. In the $\mathcal{N} = 4$ case, the vacua have similar statistics, preventing instanton tunneling.

We finally discuss the significance of these vacua, and their possible relation to Matrix theory compactification on pp-waves.

2. Matrix theory membranes

Our starting point is the result obtained by de Wit et al [1] and Kabat and Taylor [9] that a $N \to \infty$ matrix model could be used to study M2-branes of spherical topology, or "spherical supermembranes". The membrane action is[1]

$$S = -\frac{T_2}{2} \int d^3\zeta \sqrt{-g} \left(g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - 1 \right)$$

where the $\zeta^\alpha = (t, \sigma^a)$ are coordinates on the membrane worldvolume, $g_{\alpha\beta}$ is an auxiliary worldvolume metric, and $X^\mu(\zeta)$ are embedding coordinates. The light-front gauge-fixing of this action is performed in [1, 10], making use of $N \times N$ hermitean matrices to approximate the Lie algebra of area-preserving diffeomorphisms on the sphere. Specifically, consider the three cartesian coordinate functions x_1, x_2, x_3 on the unit sphere. The Poisson brackets of these functions are given by

$$\{x_A, x_B\} = \epsilon_{ABC} x_C \, .$$

From this we can produce the equations of motion for the transverse coordinates X^i:

$$\ddot{X}^i = \frac{4}{N^2} \{\{X^i, X^j\}, X^j\}$$

which follow from the hamiltonian

$$H = \frac{N}{2\pi R} \int d^2\sigma \left(\frac{1}{2} \dot{X}^i \dot{X}^i + \frac{1}{N^2} \{X^i, X^j\}\{X^i, X^j\} \right) .$$

We can then make a correspondence between these coordinate functions on S^2 with the generators of SU(2):

$$x_A \leftrightarrow \frac{2}{N} J_A$$

where J_1, J_2, J_3 are generators of the N-dimensional representation of SU(2) satisfying

$$-i[J_A, J_B] = \epsilon_{ABC} J_C \, .$$

Then completing the matrix correspondence with

$$\{\cdot, \cdot\} \to \frac{-iN}{2} [\cdot, \cdot], \qquad \frac{N}{4\pi} \int d^2\sigma \to \mathrm{Tr} \, ,$$

we produce the hamiltonian of Matrix theory,

$$H = \frac{1}{R} \mathrm{Tr} \left(\frac{1}{2} \dot{\mathbf{X}}^i \dot{\mathbf{X}}^i - \frac{1}{4} \left[\mathbf{X}^i, \mathbf{X}^j \right] \left[\mathbf{X}^i, \mathbf{X}^j \right] \right) \tag{2.1}$$

where $\mathbf{X}^i(t) = \phi_i(t) J_i$, $i = 1, 2, 3$ as the (noncommutative) i-coordinate of a supermembrane in light-cone gauge. The equations of motion are then:

$$\begin{aligned} \ddot{\phi}_1 &= -\phi_1(\phi_2^2 + \phi_3^2) \, , \\ \ddot{\phi}_2 &= -\phi_2(\phi_1^2 + \phi_3^2) \, , \\ \ddot{\phi}_3 &= -\phi_3(\phi_1^2 + \phi_2^2) \, , \end{aligned}$$

[1]The spacetime metric used initially is $\eta_{\mu\nu} = (- + \cdots +)$. Light-front coordinates are $X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{10})$. X^+ is light-front time; the conjugate energy is $p^- = \frac{1}{\sqrt{2}}(E - p^{10})$. The coordinate X^- is compactified, $X^- \simeq X^- + 2\pi R$, and the conjugate momentum $p^+ = \frac{1}{\sqrt{2}}(E + p^{10})$ is quantized, $p^+ = N/R$. The tension of the membrane is $T_2 = 1/(2\pi)^2 l_P^3$.

It was found in [9] that the membranes are unstable, and the only vacuum configuration is $\phi_1 = \phi_2 = \phi_3 = 0$. This has a clear interpretation that the tension of the brane is forcing it to contract to zero size.

In subsequent work, Myers [11] found there was a term coupling the membrane to the four-form field strength found in supergravity,

$$V_1 = \frac{i}{6R} F_{0ijk} \,\mathrm{Tr}\, \mathbf{X}^i \mathbf{X}^j \mathbf{X}^k \,. \tag{2.2}$$

As might be hoped, this can stabilize the membrane by stretching it, the membrane being charged under this gauge field. However, there was no analysis of the supersymmetry of the membrane in this background field.

At this point we should clarify what we mean by ϕ_i as the cartesian coordinates of the membranes. We define ϕ_i such that a *membrane* will have radii $\phi_i > 0$. As $\phi_i \to 0$ for one coordinate, we see the membrane is compressed along that dimension until it resembles a disk. We can then continue to $\phi_i < 0$ if we interpret it as an *antimembrane* with radius $-\phi_i$; the intuition behind this is that the membrane has now turned inside-out, and so we would expect to have opposite charge to all gauge fields (its internal orientation is now reversed). A similar situation exists if several $\phi_i < 0$, so that a membrane can be defined as $\phi_1\phi_2\phi_3 > 0$ and an antimembrane has $\phi_1\phi_2\phi_3 < 0$. In fact, many solutions which appear distinct are actually equivalent because the $\mathbf{X}^i$ are defined only up to a unitary transformation,

$$\mathbf{X}^i \to U^\dagger \mathbf{X}^i U \,.$$

For example, the two solutions

$$\phi_1, \phi_2, \phi_3 \qquad \text{and} \qquad -\phi_1, -\phi_2, \phi_3 \tag{2.3}$$

are related by $U = \exp i\pi J_3$.

3. Constructing the toy $\mathcal{N} = 2$ superpotential

The simplest model of the supermembrane would constitute 3 real bosonic fields ϕ_i representing the cartesian coordinates, depending only on time, and their fermionic partners ψ_i. Thus this is a quantum mechanical system of 3 real superfields Φ_i,

$$\Phi_i = \phi_i + i\theta\psi_i - i\psi_i^*\theta^* + \theta\theta^* D_i$$

where θ, θ^* are Grassmann variables and $D_i(t)$ is an auxiliary function. In order to produce a superpotential capable of producing V_1, we found it was necessary for a second potential to be added, equivalent to a $\mathbf{X}^i$ mass term:

$$V_2 = \frac{1}{2R} m^2 \sum_i \mathbf{X}^i \mathbf{X}^i \,.$$

As explained in [12], this matrix theory potential term is physically realized by modifying the background metric g_{++} component,

$$g_{++} = -m^2 \sum_i \mathbf{X}^i \mathbf{X}^i \,.$$

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By choosing

$$F_{0ijk} = 3m\epsilon_{ijk}$$

the entire system can be recast in the superfield formalism,

$$S = \frac{1}{R}\int d^2\theta dt \left(\sum_i |D_\theta \Phi_i|^2 - W\right) \tag{3.1}$$

$$= \frac{1}{R}\int dt \sum_i \left\{\frac{1}{2}\dot{\phi}_i\dot{\phi}_i + i\psi_i^\dagger\dot{\psi}_i - \frac{1}{2}\left|\frac{\partial W}{\partial \phi_i}\right|^2 + \sum_j \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i^\dagger\psi_j\right\}, \tag{3.2}$$

$$W = \Phi_1\Phi_2\Phi_3 - \frac{m}{2}\sum_i \Phi_i^2\,. \tag{3.3}$$

This superpotential is similar to that studied in $\mathcal{N} = 1^*$ theories [13]. In what follows we derive equations of motion from this superpotential and show that they are the same equations of motion found for the Matrix hamiltonian in the previous section. After identifying the classical vacua of this potential, we then calculate the Witten index, suggesting the breaking of supersymmetry. Finally, we describe instanton tunneling between the bosonic and fermionic vacua, confirming that supersymmetry is broken.

We first derive the potential,

$$V(\phi_i) = \frac{1}{2}\sum_{i=1}^{3}\left|\frac{\partial W}{\partial \phi_i}\right|^2 \tag{3.4}$$

$$= \frac{1}{2}\sum_{i\neq j}\phi_i^2\phi_j^2 + \frac{m^2}{2}\sum_i \phi_i^2 - 3m\phi_1\phi_2\phi_3\,. \tag{3.5}$$

The equations of motion are then

$$\begin{aligned}
\ddot{\phi}_1 &= -\phi_1\left(\phi_2^2 + \phi_3^2\right) - m^2\phi_1 + 3m\phi_2\phi_3\,,\\
\ddot{\phi}_2 &= -\phi_2\left(\phi_3^2 + \phi_1^2\right) - m^2\phi_2 + 3m\phi_3\phi_1\,,\\
\ddot{\phi}_3 &= -\phi_3\left(\phi_1^2 + \phi_2^2\right) - m^2\phi_3 + 3m\phi_1\phi_2\,.
\end{aligned} \tag{3.6}$$

These are precisely the same as the Matrix equations of motion obtained in the previous section for a spherical membrane in background fields $g_{\mu\nu}$ and F_{0ijk}, producing potentials V_1 and V_2. And because we obtained this from a superpotential, we may easily determine the supersymmetric properties of the vacuum [14].

3.1 Calculation of the Witten index

We will first locate the classical vacua satisfying $V = 0$, neglecting any quantum-mechanical tunneling which might raise the vacuum energy. From this we can determine the Witten index Δ,

$$\Delta \equiv N_+ - N_-$$

where N_+ is the number of bosonic vacua and N_- is the number of fermionic vacua. A remarkable property of the Witten index is that it does not change under deformation of parameters, so we expect this will be the index value even when tunneling effects are included.

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The superpotential has the following two distinct classical vacua, whose labeling will become apparent shortly:

$$\begin{array}{rl} & \quad\ \phi_1 \ \ \phi_2 \ \ \phi_3 \\ |F\rangle \equiv & |\ \ 0, \ \ 0, \ \ \ 0\ \ \rangle\,, \\ |B\rangle \equiv & |\ m, \ m, \ m\ \rangle\,. \end{array} \tag{3.7}$$

In listing these vacua we have neglected all physically identical vacua which may be obtained via unitary transformations, as explained previously. Note that for $|B\rangle$, $\phi_1\phi_2\phi_3 = m^3$, and so this is a membrane (as opposed to an antimembrane). Physically this arises because a certain F_{0ijk} charge is required to balance the tension and metric potentials possessing membrane-antimembrane symmetry. By definition, membranes have this charge, antimembranes having the opposite charge, so we require a membrane.

To determine the type of vacua (bosonic or fermionic) each of these are, we investigate the fermion mass term of the supersymmetric action,

$$\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} = \begin{pmatrix} -m & \phi_3 & \phi_2 \\ \phi_3 & -m & \phi_1 \\ \phi_2 & \phi_1 & -m \end{pmatrix}. \tag{3.8}$$

When the matrix is evaluated for a particular vacuum, its eigenvalues indicate the energy cost of the presence of a fermion in that ground state. A negative eigenvalue implies that the ground state contains a fermion. If an odd number of eigenvalues are negative, the ground state is fermionic. For example, the vacuum at the origin $|F\rangle$ is fermionic:

$$\lambda = \{-m,\ -m,\ -m\}\,, \tag{3.9}$$

and the extended solution $|B\rangle$ is bosonic:

$$\lambda = \{m,\ -2m,\ -2m\}\,. \tag{3.10}$$

The Witten index is therefore

$$\Delta = 1 - 1 = 0\,.$$

A non-zero Witten index implies that SUSY may or may not be broken. Although we obtained this answer by studying the vacua individually, in the $m \to \infty$ limit, the index should not change as we reduce m and "turn on" quantum mechanical interactions.

3.2 Instanton tunneling

The value of the Witten index can be confirmed and elucidated by considering the instanton tunneling that lifts certain linear combinations of these vacua. Instanton effects in supersymmetric quantum mechanics have been well-studied [15].

We seek the instanton solution with the following properties. At euclidean time $t \to -\infty$, it begins at the vacuum $|F\rangle = |0,0,0\rangle$. As time progresses it moves towards the vacuum $|B\rangle = |m,m,m\rangle$, arriving there as $t \to \infty$. Since the solution should be symmetric in all ϕ_i, we may use the following action to derive the instanton solution:

$$S_E = \frac{3}{2}\int dt\left(\dot{\phi}^2 + V(\phi) + \cdots\right) \tag{3.11}$$

where $\cdots$ indicate the fermion terms, which play no role in our calculation, and the potential is given by

$$V(\phi) = \phi^2(\phi - m)^2 .$$

The instanton solution then satisfies

$$\dot{\phi}_I = -\phi_I(\phi_I - m) .$$

Choosing t_I as the time at which the instanton is half-way between the two vacua, this is easily solved to yield

$$\phi_I(t) = \frac{m}{1 + e^{-m(t-t_I)}} . \tag{3.12}$$

Plugging this back into (3.11), we produce the classical action

$$A_0 = \frac{m^3}{2} .$$

Hence these classical vacua each acquire a mass $\sim e^{-m^3/\hbar}$.

4. Constructing the toy $\mathcal{N} = 4$ superpotential

We now extend the number of supercharges to 4 simply by complexifying the superfields Φ_i and parameter m. In particular, the superpotential (3.3) and its vacua are identical. And again, unitary transformations may be used to bring all extended solutions to the form $|m, m, m\rangle$ where the basis $|\phi_1, \phi_2, \phi_3\rangle$ is now complex. The membrane will then have radius $|m|$, with the phase of m determining the complex basis.

For the special case $m = 0$, this model reproduces the 'rotating ellipsoidal membrane' analysis in [16], where the membrane was stabilized by adding angular momentum to the complex fields. For $m \neq 0$, our potential does not have this $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ symmetry.

We may choose the convention that the six mass eigenvalues for the state $|m, m, m\rangle$,

$$\lambda = \{\operatorname{Re} m, \operatorname{Im} m, -2\operatorname{Re} m, -2\operatorname{Im} m, -2\operatorname{Re} m, -2\operatorname{Im} m\}$$

constitute a bosonic state. The point-size solution $|0, 0, 0\rangle$ now has mass eigenvalues

$$\lambda = \{-\operatorname{Re} m, -\operatorname{Im} m, -\operatorname{Re} m, -\operatorname{Im} m, -\operatorname{Re} m, -\operatorname{Im} m\} ,$$

which makes it also a bosonic state. Since there cannot be tunneling between vacua of like statistics, these vacua remain massless.

5. Discussion

We have presented a quantum mechanical model of the spherical supermembrane. For either 2 or 4 supercharges, it possesses two supersymmetric vacua, but in the former case the vacua acquire mass when instanton effects are included.

This situation is similar to that believed to occur in the study of giant gravitons [3, 4]. In fact, the instanton solution (3.12) we obtained using quantum mechanics is identical to the exact field theory result obtained in [4]. Perhaps our model could be modified to include a second extended membrane vacua, as is the case in giant gravitons.

The most interesting aspect of the model is that the background fields required for our toy model bear a striking resemblence to Matrix theory compactified on a pp-wave background with our parameter m equivalent to the parameter $\mu/3$ found in [5] (in particular, the metric and field strengths are identical). Ours differs only in that we limit ourselves to the 3 relevant bosonic fields (playing the role of the cartesian coordinates), our fermions are Grassmann numbers and carry no spinor indices arising from dimensional reduction, and our supercharges commute with the hamiltonian. It would be very interesting to determine whether our model can be obtained as a special case of supersymmetric Matrix theory compactification.

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